How to... Compute Determinants

Given: A quadratic matrix $A \in \mathbb{R}^{n\times n}$.

Wanted: The determinant $\det(A)$.

Example

We consider the Matrices $A \in \mathbb{R}^{2\times 2}$, $B \in R^{3\times 3}$, and $C \in \mathbb{R}^{4\times 4}$.

$$A = \begin{pmatrix} -1 & 7 \\ -3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 0 & 2 \\ -2 & -1 & 2 \\ 3 & -5 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 0 & -9 \\ 2 & -3 & 5 & 8 \\ -1 & 0 & 0 & 6 \\ 3 & -4 & 0 & -1 \end{pmatrix}$$

1 Use the direct formulas for $n \leq 3$

If the dimension n is smaller or equal to 3, use the following formulas.

a Case $n = 1$

$\det(a) = a$

b Case $n = 2$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ac - bd$$

c Case $n = 3$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - gec - hfa - idb$$

This computation can be easily memorized with **Sarrus' rule**:
Copy the first two columns of the matrix
The three top-left to bottom-right diagonals in this augmented matrix yield the positive terms of the determinant, the three bottom-left to top-right diagonals yield the negative terms in the determinant

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array} \qquad aei + bfg + edh - gec - hfa - idb$$

(Note, that Laplace's formula can be used for $n = 2$ and $n = 3$ as well)

Example for $n = 2$ and $n = 3$

$$\det A = \det \begin{pmatrix} -1 & 7 \\ -3 & 2 \end{pmatrix} = -1 \cdot 2 - \big((-3) \cdot 7\big) = -2 + 21 = 19$$

For $\det(B)$ we use Sarrus' rule

$$\begin{array}{ccccc} 4 & 0 & 2 & 4 & 0 \\ -2 & -1 & 2 & -2 & -1 \\ 3 & -5 & 3 & 3 & -5 \end{array}$$

and obtain

$$\begin{aligned} \det(B) &= 4 \cdot (-1) \cdot 3 + 0 \cdot 2 \cdot 3 + 2 \cdot (-2) \cdot 3 - 3 \cdot (-1) \cdot 2 - (-5) \cdot 2 \cdot 4 - 3 \cdot (-2) \cdot 2 \\ &= -12 + 0 + 10 + 6 + 40 - 0 = 44 \end{aligned}$$

2 Use Laplace's formula for $n \geq 4$

Given any matrix $A \in \mathbb{R}^{n \times n}$, denote by $A_i j \in \mathbb{R}^{n-1 \times n-1}$ the matrix obtained from A by deleting row i and column j, i.e.,

$$A_{ij} = \det \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Then choose one of the following formulas.

a Row Expansion

Choose any row i and use the following formula

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

b Column Expansion

Choose any column j and use the following formula

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

Use row- and/or column expansion until the sub-determinants $\det(A_{ij})$ can be computed with the formulas for $n \leq 3$.

Hint: As you can choose the row or column for the expansion freely, use rows with as many 0 elements as possible. This will simplify computations considerably.

Example for $n = 4$

We choose the third column for expansion. Further, we use the sign matrix on the right as a help to identify the right sign (the $(-1)^{i+j}$-term) in the formula.

$$C = \begin{pmatrix} 1 & 2 & 0 & -9 \\ 2 & -3 & 5 & 8 \\ -1 & 0 & 0 & 6 \\ 3 & -4 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Using the formula for column expansion, we obtain

$$\det C = +0 \cdot \det C_{13} - 5 \cdot \det C_{23} + 0 \cdot \det C_{33} - 0 \cdot \det C_{43}$$

We see, that we only need the submatrix C_{23} for the compuation of $\det C$. This is a 3×3-matrix, i.e., this matrix can be computed using Sarrus' rule (or using further applications of Laplace's formula). We thus get

$$\det C = -5 \cdot \det \begin{pmatrix} 1 & 2 & -9 \\ -1 & 0 & 6 \\ 3 & -4 & -1 \end{pmatrix} = (-1) + 36 + (-36) - (-27) - (-24) - 2 = -56$$